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A METHOD FOR THE OPTIMAL CONTROL OF THE MOTION OF A DYNAMIC SYSTEM IN THE PRESENCE OF CONSTANTLY OPERATING PERTURBATIONS†

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A method is proposed for the continuous correction of control in the linear problem of the optimal positional control of a dynamic system under conditions of indeterminancy [1–3], intended for use in the actual motion of microprocessors. On the assumption that the perturbations are sufficiently regular, when generating current control values, in addition to [4], analysis of the trajectory segment that has been traversed can be used. The results of computer simulations are given.

THE PROBLEM of constructing optimal feedback, formulated at the start of the 1950s, has proved to be exceptionally difficult [5] and has still not been solved, apart from special cases [1, 2, 6] and in one specific case [7, 8]. We proposed a different approach to the problem in [4], and this is extended below to the case where, in addition to the optimal regulator, the feedback includes a predictor which, from an analysis of traversed trajectory segments of the dynamical system, predicts the possible effect of a perturbation in a certain time interval in the future. On the basis of this information, the regulator generates the current values of the control action.

The proposed method of controlling a dynamic system can be effective for certain types of perturbation. If the prediction procedure is dropped, it becomes the classical optimal feedback method of control. It should be noted that when stochastic models of sufficiently regular perturbations (in the probability sense) are used, prediction is the typical operation that feedback performs [9].

1. STATEMENT OF THE PROBLEM

In the class of piece-wise continuous functions $u(t)$, $t\in T=[0, t^*]$, we consider the linear problem of the optimal control of a dynamic system

$$J(u)=h_0'x(t^*)\to\max \tag{1.1}$$

$$x^{\cdot}=Ax+bu,\quad x(0)=x_0 \tag{1.2}$$

$$Hx(t^*)=g \tag{1.3}$$

$$|u(t)|\leqslant 1,\quad t\in T\quad (x\in R^n,\ g\in R^m,\ u\in R) \tag{1.4}$$

As we know [1, 3], optimal programmed control of problem (1.1)–(1.4) is the name given to each piecewise-continuous function $u^0(\cdot)=(u^0(t),\ t\in T)$, for which the constraint $|u^0(t)|\leqslant 1$, $t\in T$ holds, which transforms the (optimal) trajectory $x^0(t)$, $t\in T$ of system (1.2) at time t^* to the terminal set $X^*=\{x\in R^n\colon Hx=g\}$, defined by (1.3), and gives a maximum value for quality criterion (1.1).

Optimal programmed controls are used, as a rule, to evaluate the potential of a dynamic system,

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but are rarely employed in practice, owing to the fact that they are incapable of allowing for the inevitable perturbations that arise during actual functioning of system (1.1)–(1.4).

It is for this reason that, in applications, preference is given to positional optimal control (feedback-type optimal controls). By this is meant any piecewise-continuous functions $u^0(x,t)$, $x\in R^n$, $t\in T$ which satisfy the inequality $|u^0(x,t)|\leqslant 1$, $x\in R^n$, $t\in T$, and for any permissible position $\{\tau, x^*(\tau)\}$ generate a solution of the system

$$x^{\cdot}=Ax+bu^0(x,t),\quad x(\tau)=x^*(\tau)$$

which coincides with the optimal trajectory for this position. Unlike programmed control $u^0(t)\in T$, $t\in T$, positional control $u^0(x,t)\in T$, $x\in R^n$, $t\in T$ is capable of reacting to many unpredicted perturbations $w(t)$, $t\in T$, that is, the trajectories $x^*(t)$, $t\in T$ of the system

$$x^{\cdot}=Ax+bu^0(x,t)+w(t),\quad t\in T,\quad x(0)=x_0 \tag{1.5}$$

are completely satisfactory from the practical point of view.

The algorithms of the operation of regulators described in [4] construct, in real time, controls $u^*(t)$, $t\in T$ that are identical with samples $u^0(x^*(t),t)$, $t\in T$ of optimal feedback control along the process $x^*(t)$, $t\in T$ satisfying Eq. (1.5) under the effect of a perturbation $w(t)$, $t\in T^0=[0,t^0]$, $0<t^0<t^*$; $w(t)\equiv 0$, $t\in]t^0, t^*]$, for which allowance was not made in (1.1)–(1.4). Possession of this property is the basic principle underlying the regulators constructed in [4]. Regulators can be constructed using other principles also. The algorithm of operation of the regulator described in this paper uses the results of an analysis of a segment of a traversed trajectory to generate the control action.

We assume that the regulator has been constructed and has been in operation for a time$T_0(\tau)=[0,\tau]$, $0<\tau<t^0$. We denote by $u^*(t)$, $t\in T_0(\tau)$ the control produced by the regulator, and by $x^*(t)$, $t\in T_0(\tau)$ the trajectory of the system that has been traversed

$$x^{\cdot *}=Ax^*+bu^*+w(t),\quad x^*(0)=x_0,\quad t\in T_0(\tau) \tag{1.6}$$

The function

$$z^w(t)=x^*(t)-z^u(t),\quad z^u(t)=F(t,0)x_0+\int_0^t F(t,s)bu^*(s)\,ds$$

$$t\in T,\quad (F^{\cdot}(t)=AF(t),\quad F(0)=E,\quad F(t,\tau)=F(t)F^{-1}(\tau))$$

characterizes the drift of the trajectory (1.6) due to the effect of the perturbation $w((t)$, $t\in T$. We assume that perturbations $w(t)$, $t\in T$ are continuous and fairly regular and behave according to certain (unknown) relations, by virtue of which the functions $z^w(t)$, $t\in T^0$ can be approximated quite well in the given finite-parametric class of functions

$$z(\cdot,v)=(z(t,v),\ t\in T^0)$$
$$v\in V=\{v\in R^r\colon d_*\leqslant v\leqslant d^*\},\quad r\geqslant 2n$$

On this basis, at a time $\tau>0$ we consider the segment

$$z_\tau^w(\cdot)=(z^w(t),\ \mu(\tau)\leqslant t\leqslant\tau),\quad 0\leqslant\mu(\tau)<\tau$$

of a trajectory that has drifted, and approximate it by the function $z(t,v(\tau))$, $\mu(\tau)\leqslant t\leqslant\tau$ from the finite-parametric family mentioned above

$$\begin{aligned}\nu(\tau)&=\max_{i=1,2,\dots,n}\ \max_{t\in[\mu(\tau),\tau]}|z_i^w(t)-z_i(t,v(\tau))|=\\ &=\min_{v}\ \max_{i=1,2,\dots,n}\ \max_{t\in[\mu(\tau),\tau]}|z_i^w(t)-z_i(t,v)|\\ &v\in V,\quad z^w(\tau)=z(\tau,v),\quad \dot z^w(\mu(\tau))=z(\mu(\tau),v)\end{aligned} \tag{1.7}$$

We select the interval $[\tau,\lambda(\tau)]$, $\tau\leqslant\lambda(\tau)\leqslant t^0$ and assume that drift of the trajectory $z^w(t)$, $t\in[\tau,\lambda(\tau)]$, in this interval is the same as the function $z(t,v(\tau))$, $t\in[\tau,\lambda(\tau)]$. We take $\mu(t)$ and $\lambda(t)$ to be certain continuous known functions of τ: $0\leqslant\mu(\tau)\leqslant\tau\leqslant\lambda(\tau)\leqslant t^0$.

The piecewise-continuous function $u_\tau(t)=u(t|\tau,\ x^*(\tau),\ v(\tau))$, $t\in T_\tau=[\tau,t^*]$ satisfying the constraint (4) will be called a τ-permissible control if it corresponds to a trajectory $x_\tau(t)=x(t|\tau,$

$x^*(\tau), v(\tau)), t\in T_\tau$ of system (1.2) which, at time τ, emerges from the state $x^*(\tau)$, experiences drift $z^w(t) = z(t, v(\tau))$ in the interval $[\tau, \lambda(\tau)]$, and at time t^* reaches the set X^*; the τ-permissible control $u_\tau^0(t) = u^0(t|\tau, x^*(\tau), v(\tau)), t\in T_\tau$ is said to be τ-optimal if it gives a maximum of the functional $J(u) = h_0'x(t^*)$: $J(u_\tau^0) = \max J(u_\tau)$.

We shall call the piecewise-continuous function $u^0(t, x, v), t\in T, x\in R^n, v\in V$ an optimal control of feedback type (optimal positional control) if the function

$$\bar{x}(t) = \begin{cases} x(t) + z(t, v) - F(t, \tau)z(\tau, v), & t\in[\tau, \lambda(\tau)] \\ x(t) + F(t, \lambda(\tau))z(\lambda(\tau), v) - F(t, \tau)z(\tau, v), & t\in\,]\lambda(\tau), t^*] \end{cases}$$

(which represents the result of imposing a drift $z^w(t) = z(t, v)$ on the trajectory $x(t), t\in T_\tau$ of the system

$$x^{\cdot} = Ax + bu^0(t, x, v), \quad x(\tau) = x^*$$

in the interval $t\in[\tau, \lambda(\tau)]$) for any τ, x^*, v from the range of controllability.

If the optimizing system and mathematical model (1.2), which under actual conditions experiences the effect of unknown perturbations $w(t), t\in T$, is closed by feedback $u^0(t, x, v)$, then its behaviour will be described by the equation

$$x^{\cdot} = Ax + bu^0(t, x(t), v(t)) + w, \quad x(0) = x_0 \tag{1.8}$$

where $v(t)$ is the solution of (1.7) when $\tau = t\in T^0$.

We denote by $w^*(t), t\in T^0$; $w^*(t)\equiv 0, t^0<t\leqslant t^*$ a perturbation that occurs in the given process. It will correspond to a trajectory $x^*(t), t\in T$, Eqs (1.8), the function $v^*(t), t\in T$ of (1.7) and control $u^*(t) = u^0(t, x^*(t), v^*(t)), t\in T$, which circulates in the closed system (1.7), (1.8).

A unit which generates the control $u^*(t), t\in T$ in real time in each specific part of the functioning of the system is called an optimal regulator.

This definition of a regulator does not assume that the optimal feedback control $u^0(t, x, v), t\in T, x\in R^n, v\in V$ is known.

The purpose of the present investigation is to describe an algorithm for the operation of the regulator (Sec. 4). First, in Secs 2 and 3, we describe algorithms for constructing in real time the τ-optimal control $u_\tau^0(\cdot)$ and the function $v(\tau), \tau\in T^0$.

Note. 1. An operator of a different kind can be obtained if drift $z^w(t)$ of the trajectory in the interval $t\in[\tau, \lambda(\tau)]$ is assumed to be a certain function from a $\nu(t)$-neighbourhood of the function $z(t, v(\tau)), t\in[\tau, \lambda(\tau)]$. In that case, the guaranteed result principle must naturally be used in the definition of τ-permissible and τ-optimal control.

2. When constructing the approximation $z(t, v(\tau)), t\in[\mu(\tau), \tau]$ of function $z^w(t), t\in[\mu(\tau), \tau]$, a requirement that can be added to $z^w(\tau) = z(\tau, v), z^w(\mu(\tau)) = z(\mu(\tau), v)$ is that the two functions shall be identical at several points in $[\mu(\tau)\ \tau]$.

2. THE DEFINING EQUATIONS OF THE REGULATOR

From the definition of τ-optimal control $u_\tau^0(\cdot) = (u_\tau^0(t), t\in T_\tau)$, it follows that it is a solution of the system

$$h_0'x(t^*)\to\max, \quad x^{\cdot} = Ax + bu, \quad x(\tau) = 0 \tag{2.1}$$
$$Hx(t^*) = g(\tau); \quad |u(t)|\leqslant 1, \quad t\in T_\tau$$

where

$$g(\tau) = g - H[F(t^*, \tau)(x^*(\tau) - z(\tau, v(\tau))) + F(t^*, \lambda(\tau))z(\lambda(\tau), v(\tau))] \tag{2.2}$$

We will describe the real time construction algorithm for the solution $u_\tau^0(\cdot)$ of problem (2.1) for

$\tau \in T^0 = [0, t^0]$ on the assumption that $g(\tau)$ is a certain known m-vector function which, together with the given time t^0, possesses the following property

$$\text{rank}(HF(t^*, t)b, \ t \in T_*(\tau)) = m, \quad \forall \tau \in T^0$$

where $T_*(\tau)$ are points of discontinuity of the function $u_\tau^0(t) = u^0(t|\tau, g(\tau))$, $t \in]\tau, t^*]$. The solution $u_\tau^0(\cdot)$ of (2.1) has the form [10]

$$u_\tau^0(t) = \text{sign}\, \psi_\tau'(t) b, \quad t \in T_\tau \tag{2.3}$$

where $\psi_\tau(t)$, $t \in T_\tau$, is a solution of the conjugate system $\dot{\psi} = -A'\psi$, $\psi(t^*) = h_0 - H'y(\tau)$ and $y(\tau) = (y_i(\tau), i = 1, 2, \ldots, m)$ is the optimal potential vector.

From (2.3) it follows that the solution of (2.1) is defined by the set

$$t_i(\tau), \ i = 1, 2, \ldots, p(\tau); \ \ y(\tau)$$

consisting of points $t_i(\tau)$, $i = 1, 2, \ldots, p(\tau)$ at which the function $\Delta_\tau(t) = \psi_\tau'(t)b$, $t \in T_\tau$, becomes zero, together with the optimal potential vector. (We assume that $t_i(\tau) < t_{i+1}(\tau)$, $i = 1, 2, \ldots, p(\tau) - 1$.)

We construct functions $k_i(\tau)$, $i = 1, 2, \ldots, p(\tau)$

$$\begin{aligned} &k_i(\tau) = -1, \ \text{if} \ \ \Delta_\tau(t) < 0, \ \ t \in]t_i(\tau), t_{i+1}(\tau)[\\ &k_i(\tau) = 1, \ \text{if} \ \ \Delta_\tau(t) > 0, \ \ t \in]t_i(\tau), t_{i+1}(\tau)[\\ &i = 0, 1, \ldots, p(\tau), \ \ (t_0(\tau) = \tau, \ t_{p(\tau)+1}(\tau) = t^*) \end{aligned}$$

Suppose that at $\tau = \tau_0$ the solution of (2.1) satisfies the relations

$$\begin{aligned} &\tau < t_1(\tau), \quad t_{p(\tau)} < t^*; \quad \partial \Delta_\tau(t)/\partial t|_{t = t_i(\tau)} \neq 0 \\ &i = 1, 2, \ldots, p(\tau) \end{aligned} \tag{2.4}$$

It is obvious that the functions $p(\tau)$, $k_i(\tau)$, $i = 0, 1, \ldots, p(\tau)$, $\tau \in T^0$ are piecewise-constant (discrete). Allowing for this and the fact that relations (2.4) hold at $\tau = \tau_0$, we have

$$\begin{aligned} &p(\tau) = p(\tau_0) = p, \quad k_i(\tau) = k_i(\tau_0) = k_i \\ &i = 0, 1, \ldots, p; \quad \tau \in T^+(\tau_0) \end{aligned} \tag{2.5}$$

Here $T^+(\tau_0)$ is a sufficiently small right-hand neighbourhood of the point τ_0.

Then, from (2.3) and (2.4), we conclude that the continuous functions

$$t_i(\tau), \quad i = 1, 2, \ldots, p; \ \ y(\tau) \tag{2.6}$$

which define a solution of (2.1) are uniquely defined by the system of equations

$$\begin{aligned} &f(\tau; t_i(\tau), \ \ i = 1, 2, \ldots, p; \ \ g(\tau)) = 0 \\ &q_j(t_i(\tau), \ i = 1, 2, \ldots, p; \ \ y(\tau)) = 0, \quad j = 1, 2, \ldots, p \end{aligned} \tag{2.7}$$

where

$$f(\tau; t_i, \ i = 1, 2, \ldots, p; \ g) = \sum_{i=0}^{p} k_i \int_{t_i}^{t_{i+1}} HF(t^*, t) b\, dt - g$$

$$q_j(t_i, \ i = 1, 2, \ldots, p; \ y) = (h_0' - y'H) F(t^*, t_j) b, \qquad j = 1, 2, \ldots, p$$

$$t_0 = \tau, \quad t_{p+1} = t^*$$

The equations of (2.7) are called the defining equations of the regulator.

A numerical method of constructing solution (2.6) of Eqs (2.7) for regions where the functions (2.5) are constant, the rules for finding the points of discontinuity $\tau \in T^0$ of the functions (2.5) and the rules for joining the solutions (2.6) of Eqs (2.7) at time τ were given before in [4].

It is clear from (2.7) that the behaviour of the set (2.6) depends on the function $g(\tau)$ which, in turn, is constructed using (2.2) and depends on the solution of (1.7).

Before going on to describe the algorithm of the operation of the regulator, we will describe the algorithm of operation of a unit which generates the solution $v(\tau)$ of problem (1.7) in real time. We will call this unit a predictor.

3. ALGORITHM OF THE OPERATION OF A PREDICTOR

Suppose that

$$z(t, v) = \sum_{j=1}^{r} v_j \omega_j(t), \quad t \in T$$

where $\omega_j(t) = (\omega_{ij}(t), i = 1, 2, \ldots, n), j = 1, 2, \ldots, r, t \in T$ are known functions. Then (1.7) can be written in the form

$$\begin{aligned} &\nu \to \min \\ &-\nu \leqslant z_i^w(t) - \sum_{j=1}^{r} v_j \omega_{ij}(t) \leqslant \nu \\ &i = 1, 2, \ldots, n; \quad t \in T(\tau) = [\mu(\tau), \tau] \\ &z^w(\tau) = \sum_{j=1}^{r} v_j \omega_j(\tau); \quad z^w(\mu(\tau)) = \sum_{j=1}^{r} v_j \omega_j(\mu(\tau)), \quad v \in V \end{aligned} \tag{3.1}$$

Let $(\nu(\tau), v(\tau))$, $v(\tau) = (v_j(\tau), j = 1, 2, \ldots, r)$ be the optimal plan of problem (3.1). We put

$$\begin{aligned} &J = \{1, 2, \ldots, r\}, \quad I = \{1, 2, \ldots, n\} \\ &Q(\tau) = \{(i, t) \in I \times T(\tau): \; |z_i^w(t) - \sum_{j \in J} v_j(\tau)\omega_{ij}(t)| = \nu(\tau)\} = \\ &= \{(i_k(\tau), t_k(\tau)), k \in K(\tau)\} \\ &q_k(\tau) = \operatorname{sign}(z_{i_k}^w(t_k) - \sum_{j \in J} v_j(\tau)\omega_{i_k j}(t_k)), \quad k \in K(\tau) \\ &J_\tau^+ = \{j \in J: \; v_j(\tau) = d_j^*\}, \quad J_\tau^- = \{j \in J: \; v_j(\tau) = d_{*j}\} \end{aligned} \tag{3.2}$$

Here $K(\tau)$ is a finite subset of the set of natural numbers.

By analogy with the known result [11], it can be shown that there exist n-vectors $\eta^*(\tau)$, $\eta_*(\tau)$ and numbers $y_k(\tau)$, $k \in K(\tau)$

$$q_k(\tau) y_k(\tau) \leqslant 0, \quad k \in K(\tau); \quad \sum_{i \in K(\tau)} |y_k(\tau)| = 1$$

such that for $v(\tau)$ and estimates

$$\Delta_j(\tau) = \sum_{k \in K(\tau)} \omega_{i_k j}(t_k(\tau)) y_k(\tau) + \eta^{*\prime}(\tau)\omega_j(\tau) + \eta_*'(\tau)\omega_j(\mu(\tau)), \quad j \in J$$

the relations

$$\begin{aligned} &\Delta_j(\tau) \geqslant 0 \quad \text{for} \quad j \in J_\tau^-; \quad \Delta_j(\tau) \leqslant 0 \quad \text{for} \quad j \in J_\tau^+ \\ &\Delta_j(\tau) = 0, \quad j \in J \setminus (J_\tau^+ \cup J_\tau^-) \end{aligned}$$

are satisfied.

We will call the vector $\xi(\tau) = (\eta^*(\tau), \eta_*(\tau); y_k(\tau), k \in K(\tau))$ an optimal dual plan. The set $\{\nu(\tau), v(\tau), \xi(\tau)\}$ of direct optimal dual plans will be called a solution of problem (3.1).

The solution of (3.1) is non-degenerate if

$$\operatorname{rank}\left\| \begin{matrix} B_\tau(K(\tau), J_\tau^0) \\ C_\tau(K(\tau), J_\tau^0) \end{matrix} \right\| = |J_\tau^0| + 1, \quad d_{*j} < v_j(\tau) < d_j^*, \quad j \in J_\tau^0 \tag{3.3}$$

$$\operatorname{rank} B_\tau(K(\tau), J_\tau^0) = |K(\tau)| + 2n, \quad y_k(\tau) \neq 0, \quad k \in K(\tau) \tag{3.4}$$

Here

$$J_\tau^0 = \{ j \in J\colon \ \Delta_j(\tau) = 0 \}; \quad C_\tau(K_*, J_*) = \left\| \begin{matrix} \dot\omega_{i_k j}(t_k(\tau)), \ j \in J_*; 0 \\ k \in K_* \end{matrix} \right\|$$

$$B_\tau(K_*, J_*) = \left\| \begin{matrix} \omega_{i_k j}(t_k(\tau)), \ j \in J_*; q_k(\tau) \\ k \in K_* \\ \hline \omega_j(\tau), \quad j \in J_*; \quad 0 \\ \omega_j(\mu(\tau)), \ j \in J_*; \quad 0 \end{matrix} \right\|$$

Let $\{\nu(\tau^0),\ v(\tau_0);\ \xi(\tau_0)\}$ be a non-degenerate solution of (3.1) for $\tau = \tau_0$. We will describe an algorithm of the operation of a predictor which, in real time, constructs a solution $\{\nu(\tau),\ v(\tau);\ \xi(\tau)\}$ of problem (3.1) for $\tau \in]\tau_0, t^0]$ on the assumption that the function $v(\tau)$ is continuous in the interval $[\tau_0, t^0]$.

Let $T^+(\tau_0)$ denote a sufficiently small right-hand neighbourhood of the point τ_0.

Consider the set of functions

$$\begin{gathered} \nu(\tau), v(\tau); \quad y_k(\tau), \ k \in K(\tau); \quad \eta^*(\tau), \eta_*(\tau) \\ i_k(\tau), t_k(\tau), q_k(\tau), \ k \in K(\tau); \quad J_\tau^+, J_\tau^- \end{gathered} \tag{3.5}$$

which define a solution of (3.1) for $\tau \geqslant \tau_0$. From these, we select piecewise-constant (discrete) functions

$$J_\tau^+, J_\tau^-, K(\tau); \quad i_k(\tau), q_k(\tau), \ k \in K(\tau) \tag{3.6}$$

and continuous functions

$$\nu(\tau), v(\tau); \quad y_k(\tau), \ t_k(\tau), \ k \in K(\tau); \quad \eta^*(\tau), \ \eta_*(\tau) \tag{3.7}$$

Since the solution of (3.1) at $\tau = \tau_0$ is non-degenerate, for discrete functions (3.6) we have

$$\begin{gathered} J_\tau^+ = J_{\tau_0}^+ = J^+, \quad J_\tau^- = J_{\tau_0}^- = J^-, \quad K(\tau) = K(\tau_0) = K \\ i_k(\tau) = i_k(\tau_0) = i_k, \quad q_k(\tau) = q_k(\tau_0) = q_k, \quad k \in K, \quad \tau \in T^+(\tau_0) \end{gathered} \tag{3.8}$$

From (3.2) and (3.8) it follows that a unique set of continuous functions (3.7) consisting of the solution of (3.1) and active times can be found for $\tau \in T^+(\tau_0)$ from the equations

$$\begin{gathered} v_j(\tau) = d_j^*, \quad j \in J^+; \quad v_j(\tau) = d_{*j}, \quad j \in J^- \\ f_{1k}(v(\tau), \nu(\tau), t_k(\tau)) = 0, \quad f_{2k}(v(\tau), t_k(\tau)) = 0, \quad k \in K \\ q^*(v(\tau), \tau) = 0, \quad q_*(v(\tau), \tau) = 0, \quad \Delta_*(y_k(\tau), k \in K) = 0 \\ \Delta_j(t_k(\tau), y_k(\tau), k \in K; \ \eta^*(\tau), \eta_*(\tau), \tau) = 0, \quad j \in J^0 \end{gathered} \tag{3.9}$$

Here

$$f_{1k}(v, \nu, t) = \sum_{j=1}^r v_j \omega_{i_k j}(t) - z_{i_k}^w(t) + q_k \nu$$

$$f_{2k}(v, t) = \sum_{j=1}^r v_j \dot\omega_{i_k j}(t) - \dot z_{i_k}^w(t), \quad k \in K$$

$$\Delta_*(y_k, k \in K) = \sum_{k \in K} q_k y_k + 1; \quad q^*(v, \tau) = \sum_{j=1}^r v_j \omega_j(\tau) - z^w(\tau)$$

$$q_*(v,\tau)=\sum_{j=1}^{r}v_j\omega_j(\mu(\tau))-z^w(\mu(\tau))$$

$$\Delta_j(t_k,y_k,k\in K;\eta^*,\eta_*,\tau)=\sum_{k\in K}\omega_{i_kj}(t_k)y_k+$$
$$+\eta^{*\prime}\omega_j(\tau)+\eta_*'\omega_j(\mu(\tau)),\quad j\in J^0=J\setminus(J^+\cup J^-)$$

Hence, for $\tau\geqslant\tau_0$ the functions (3.7) [the solution of (3.1)] can be constructed as follows.

The interval $[\tau_0, t^0]$ is divided into segments in which the discrete functions (3.6) are constant. On each segment, the continuous functions (3.7) are found from Eqs (3.9), by the numerical method given in Sec. 5. The time $\bar\tau$ of discontinuity of functions (3.6) is characterized by one of the following properties:

1. for some $(i_0, t_0)\in I\times T(\bar\tau)\setminus\{(i_k, t_k(\bar\tau), k\in K\}$

$$\sum_{j\in J}\omega_{i_0j}(t_0)v_j(\bar\tau)=z_{t_0}^w(t_0)-q_0\nu(\bar\tau)$$

when $q_0=1$ or $q_0=-1$;

2. for a certain $j_0\in J^0$ either $v_{j_0}(\bar\tau)=d_{*j_0}$ or $v_{j_0}(\bar\tau)=d_{j_0}^*$;
3. for a certain $s_0\in K$, $y_{s_0}(\bar\tau)=0$;
4. for a certain $j_0\in J\setminus J^0$, $\Delta_{j_0}(\bar\tau)=0$.

We continue by investigating the general case, thereby excluding cases where (a) two or more of the above properties 1–4 are satisfied simultaneously, (b) any of properties 1–4 hold for several $j\in J$ or several j and times $(i,t)\in I\times T(\bar\tau)$, and (c) for a certain $(i,t)\in I\times T(\tau)$

$$\Big|\sum_{j\in J}v_j(\bar\tau)\omega_{ij}(t)-z_i^w(t)\Big|=\nu(\bar\tau)$$

$$\sum_{j\in J}v_j(\bar\tau)\omega_{ij}^{(p)}(t)=z_i^{w(p)}(t),\quad p=1,2$$

Rules can be obtained [4, 11] for finding new values of the functions (3.6)

$$\bar J^+,\ \bar J^-,\ \bar K;\quad \bar i_k,\ \bar q_k,\ k\in\bar K$$

in a new interval $[\bar\tau, \tau_*]$ where they are constant, and a rule for joining the solutions of equations of the type (3.9) at times $\bar\tau$, τ_*, etc.

There are several possible ways of starting the predictor at time $\tau=0$. For example, we could proceed as follows.

Before the start of operation of the predictor, in a small initial segment $[0, h_*]$, $h_*>0$, we define a finite set of possible drifts $z^{wk}(t)$, $t\in[0,h_*]$, $k=1,2,\ldots,N$. For each drift $z^{wk}(t)$, $t\in[0,h_*]$, we find the solution (ν^k, v^k) of problem (1.7) for $\tau=h_*$, $\mu(h_*)=0$. Then after switching on the predictor, we measure the true value of the drift that has occurred in $[0,h_*]$. We find the drift $z^{wk_0}(t)$, $t\in[0,h_*]$ closest to it. We reduce the corresponding vector (ν^{k_0}, v^{k_0}) to the solution $(\nu(h_*), v(h_*))$ of problem (1.7) at $\tau=h_*$ by a procedure of the type described in [11].

4. AN ALGORITHM OF THE OPERATION OF THE REGULATOR

We choose the parameter $\epsilon>0$ which characterizes the limiting switching frequently generated by the control regulator. We fix the functions $\mu(t)$ and $\lambda(t)$ (assuming that the functions $\omega_j(t)$, $t\in T^0$, $j=1,2,\ldots,r$ are known). At the initial time $\tau=0$, the regulator selects the value

$$u^*(0)=u_{\tau=0}^0(0)$$

that is, it starts operation with optimal programmed control. For any current time τ, the regulator selects the control value by the rule

$$u^*(\tau)=\begin{cases} u_\tau^0(\tau), & \tau-\mathbf{t}(\tau)\geqslant\epsilon \\ u^*(\tau-0), & \tau-\mathbf{t}(\tau)<\epsilon \end{cases}$$

where $\mathbf{t}(\tau)$ is the nearest point of discontinuity to τ of control $u^*(t)$, $t\in[0,\ \tau[$, $\mathbf{t}(0)=-\infty$; $u_\tau^0(t)$, $t\in T_\tau$ is the solution of (2.1) obtained when solving the defining equations (2.7) using the function $g(\tau)$ (2.2) in which the vector $v(\tau)$ is found from the defining equations of the predictor (3.9).

After operating as described, the regulator constructs relay control $u^*(t)$, $t\in T$; $|u^*(t)|=1$, $t\in T$, the switching points of which on the set T^0 are not less than a distance ϵ apart.

5. A NUMERICAL METHOD OF CONSTRUCTING THE SOLUTION OF EQS (3.9)

We construct matrices

$$G(v;t_k,\ y_k,\ k\in K;\tau)=\begin{Vmatrix} B & D_1 & 0 \\ C & D_2 & 0 \\ 0 & \bar{C}' & B' \end{Vmatrix},\quad D_1=\begin{Vmatrix} D_0 \\ 0 \end{Vmatrix}$$

$$B=\begin{Vmatrix} \omega_{i_kj}(t_k),\ j\in J^0,\ q_k \\ k\in K \\ ------------ \\ \omega_j(\tau),\ j\in J^0;\quad 0 \\ \omega_j(\mu(\tau)),\ j\in J^0;\ 0 \end{Vmatrix},\quad C=\begin{Vmatrix} \omega\dot{}_{i_kj}(t_k),\ j\in J^0;\ 0 \\ k\in K \end{Vmatrix}$$

$$\bar{C}=C\operatorname{diag}(y_k,\ k\in K)$$

$$D_0=\operatorname{diag}(\alpha_k,\ k\in K),\quad D_2=\operatorname{diag}(\beta_k,\ k\in K)$$

$$\alpha_k=\sum_{j=1}^{r}v_j\dot{\omega}_{i_kj}(t_k)-\dot{z}_{i_k}^{w}(t_k)$$

$$\beta_k=\sum_{j=1}^{r}v_j\ddot{\omega}_{i_kj}(t_k)-\ddot{z}_{i_k}^{\omega}(t_k),\ k\in K$$

It can be shown that if condition (3.3), (3.4) holds,

$$\det G(v(\tau_0);\ t_k(\tau_0),\ y_k(\tau_0),\ k\in K;\ \tau_0)\neq 0$$

Thus, for $\tau\in T^+(\tau_0)$ there exists a unique solution (3.7) of system (3.9).

We will describe a numerical method of constructing this solution. Suppose that the solution (3.7) is known at nodes of the grid $\tau=\tau_0+sh$, $s=0,\ 1,\ \ldots,\ p-1$, where $h>0$ is the pitch of the grid (a parameter of the method). To compute elements (3.7) at node $\tau_{(*)}=\tau_0+ph$ we put

$$v_j^l=d_{*j},\ j\in J^-;\quad v_j^l=d_j^*,\ j\in J^+,\quad l=1,2,\ldots,l_0$$

and construct the vectors

$$z^l=(v_j^l,\ j\in J^0;\ v^l;\ t_k^l,\ k\in K;\ y_k^l,\ k\in K;\ \eta^{*l},\ \eta_*^l)$$

$$l=1,2,\ldots,l_0:$$

$$z^1=(v_j^1=v_j(\tau_{(*)}-h),\ j\in J^0;\ v^1=v(\tau_{(*)}-h);\ t_k^1=t_k(\tau_{(*)}-h),\ k\in K$$

$$y_k^1=y_k(\tau_{(*)}-h),\ k\in K;\ \eta^{*1}=\eta^*(\tau_{(*)}-h),\ \eta_*^1=\eta_*(\tau_{(*)}-h))$$

$$z^{l+1}=z^l-G^{-1}(v^l;\ t_k^l,\ y_k^l,\ k\in K,\ \tau_{(*)})(f_{1k}(v^l,\ v^l,\ t_k^l),\ k\in K;\ q^{*\prime}(v^l,\ \tau_{(*)}),\ q_*'(v^l,\ \tau_{(*)})$$

$$f_{2k}(v^l,\ t_k^l),\ k\in K;\ \Delta_j(t_k^l,\ y_k^l,\ k\in K;\ \eta^{*l},\ \eta_*^l),\ \ j\in J^0$$

$$\Delta_*(y_k^l,\ k\in K))'$$

Here $l_0=l_0(h)$ is a natural number (a parameter of the method).

We put

$$v(\tau_{(*)})=v^{l_0},\quad v(\tau_{(*)})=v^{l_0};\quad t_k(\tau_{(*)})=t_k^{l_0}$$

$$y_k(\tau_{(*)})=y_k^{l_0},\ k\in K;\quad \eta^*(\tau_{(*)})=\eta^{*l_0},\quad \eta_*(\tau_{(*)})=\eta_*^{l_0}$$

The method can be modified in various ways, but this is beyond the scope of the present paper.

6. EXAMPLE

We will illustrate the results given above on the problem of optional control of oscillatory motion

$$\int_0^{4\pi} u(t)\,dt \to \min, \quad x^{\cdot\cdot} + x = u, \quad x(0) = 0{,}703$$

$$x^{\cdot}(0) = -0{,}955; \quad x(4\pi) = x^{\cdot}(4\pi) = 0; \quad 0 \leqslant u(t) \leqslant 1, \quad t \in [0;4\pi]$$

In this problem, optimal programmed control is of the relay type. It takes only two values: 0 and 1, the switching times $t_1 = 0.334$; $t_2 = 0.936$; $t_3 = 6.62$; $t_4 = 7.22$, with $u^0(t) = 0$ in $[0; t_1]$. The quality criterion on optimal programmed control is equal to $J(u^0) = 1.2038$.

Suppose that, as the given system functions, a perturbation $w^*(t) = 0.1\sin 5t$, $t \in [0; 5]$; $w^*(t) = 0$, $t \in]5; 4\pi]$ operates which was not take into account in the model and is not known to the regulator.

The optimal relay regulator [4] generates the control $u^*(t)$, $t \in T$ with switching points $t_1 = 0.34$; $t_2 = 0.99$; $t_3 = 6.57$; $t_4 = 7.14$, with $u^*(t) = 0$, $t \in [0; t_1[$. On control $u^*(t)$, $t \in T$, the quality criterion takes the value $J(u^*) = 1.1813$.

If the regulator knew the perturbation beforehand, the corresponding optimal programmed control would have switching points $t_1 = 0.333$; $t_2 = 0.923$; $t_3 = 6.62$; $t_4 = 7.21$ and $J = 1.1794$.

At each current time τ, suppose that the regulator knows the value of perturbation $w^*(t)$ in the interval $[\tau, \lambda(\tau)]$ ($\lambda(\tau) = t + 0.3$, $t \in [0; 4.7]$, $\lambda(t) = 5$, $t \in]4.7; 5]$. Then the regulator will construct the control $u^*(t)$, $t \in T$ with switching points $t_1 = 0.36$; $t_2 = 0.975$; $t_3 = 6.586$; $t_4 = 7.151$, with $J(u^*) = 1.1806$.

We switch the predictor into feedback mode. To demonstrate the part played by the predictor in improving the efficiency of control, we shall confine ourselves to a special non-optimal modification. We shall assume that the predictor approximates each drift component by a second-order polynomial

$$z_i(t, v) = v_2^{(i)} t^2 + v_1^{(i)} t + v_0^{(i)}, \quad i = 1, 2$$

and from these chooses two which are equal to the components of drift that took place at points τ, $\mu(\tau)$ and have equal derivatives to those components at the point τ. We choose the function $\mu(t)$, $t \in [0; 5]$ in the form $\mu(t) = 0$, $t \in [0; 0.3]$; $\mu(t) = t - 0.3$, $t \in]0.3; 5]$. The required functions $v_j^{(i)}(\tau)$, $\tau \in [0, t^0]$, $j = 0, 1, 2$; $i = 1, 2$, have the form

$$v_2^{(i)}(\tau) = [z_i^{\cdot w}(\tau) - (z_i^w(\tau) - z_i^w(\mu(\tau)))/(\tau - \mu(\tau))]/(\tau - \mu(\tau))$$

$$v_1^{(i)}(\tau) = (z_i^w(\tau) - z_i^w(\mu(\tau)))/(\tau - \mu(\tau)) - v_2^{(i)}(\tau)\,(\tau - \mu(\tau))$$

$$v_0^{(i)}(\tau) = z_i^w(\mu(\tau)) - v_2^{(i)}(\tau)\mu^2(\tau) - v_1^{(i)}(\tau)\mu(\tau)$$

$$z^w(t) = (z_i^w,\ i = 1, 2) = x^*(t) - F(t, 0)x_0 - \int_0^t F(t, s)bu^*(s)\,ds, \quad t \in T$$

The control $u^*(t)$, $t \in T$, selected by the regulator using the results of the operation of the predictor, has switching points $t_1 = 0.4$; $t_2 = 0.955$; $t_3 = 6.554$; $t_4 = 7.180$, with $u^*(t) = 0$, $t \in [0; t_1[$. The quality criterion on it takes the value $J(u^*) = 1.1808$.

The above calculations show that the result obtained by an optimal regulator [4] is improved by 19×10^{-4} if there is complete knowledge of a future perturbation. Accurate knowledge of a future perturbation in a preceding segment of length 0.3 yields a gain of 8×10^{-4} units (42% of the maximum possible). The (non-optimal) use of a predictor with respect to a perturbation in an interval of length 0.3 gave a gain of 6×10^{-5}, i.e. 75% of the previous result.

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